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On the next-to-leading-order correction to the effective action in $N = 2$ gauge theories [1]

Sergei V. Ketov [2]

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
ketov@itp.uni-hannover.de

Abstract

I attempt to analyse the next-to-leading-order non-holomorphic contribution to the Wilsonian low-energy effective action in the four-dimensional $N = 2$ gauge theories with matter, from the manifestly $N = 2$ supersymmeric point of view, by using the harmonic superspace. The perturbative one-loop correction is found to be in agreement with the $N = 1$ superfield calculations of de Wit, Grisaru and Roček. The previously unknown coefficient in front of this non-holomorphic correction is calculated. A special attention is devoted to the $N = 2$ *superconformal* gauge theories, whose one-loop non-holomorphic contribution is likely to be *exact*, even non-perturbatively. This leading (one-loop) non-holomorphic contribution to the LEEA of the $N = 2$ superconformally invariant gauge field theories is calculated, and it does not vanish, similarly to the case of the $N = 4$ super-Yang-Mills theory.

1   Introduction

Extended supersymmetry severely restricts the form of the quantum effective action in the four-dimensional gauge theories like $N = 2$ QCD. The Wilsonian effective action to be obtained by integrating out all massive degrees of freedom from the fundamental (microscopic) Lagrangian is highly non-local, but it can be expanded in powers of space-time momenta divided by the characteristic physical scale Λ. Under certain physical assumptions about the global structure of the quantum moduli space of vacua, it becomes possible to obtain the exact solution to the leading low-energy contribution describing the spectrum and the *holomorphic* static gauge couplings in the Coulomb branch of the full quantum theory [1]. The natural step further is to determine the next-to-leading-order contribution to the *low-energy effective action* (LEEA), that describes *non-holomorphic* static gauge couplings. It seems to be a more complicated problem since the electromagneitc duality and holomorphy alone are not enough to fix non-holomorphic couplings.

It is therefore desirable to understand the nature of both holomorphic and non-holomorphic contributions to the LEEA from the fully $N = 2$ supersymmetric point of view, when all the relevant symmetries of the microscopic Lagrangian are manifest. It is of particular importance to explicitly calculate quantum perturbative corrections by using manifestly $N = 2$ supersymmetric Feynman rules. It would then allow one to check various proposals based on quantum calculations in components or $N = 1$ superfields. The only known approach that provides an off-shell (model-independent) formulation for both $N = 2$ gauge multiplets and hypermultiplets, as well as the manifestly $N = 2$ supersymmetric Feynman rules is the $N = 2$ *harmonic superspace* (HSS) [2, 3].

The LEEA in the gauge sector of $N = 2$ theories in HSS was recently studied in refs. [4, 5], where it was shown that the one-loop holomorphic contributions only emerge after accounting non-vanishing central charges in the $N = 2$ SUSY algebra. It was also demonstrated [4] that the one-loop holomorphic contribution in fact coincides with Seiberg's perturbative LEEA which was originally obtained by integrating the chiral anomaly [6]. The same central charges also play an important role in the matter hypermultiplet sector of LEEA. As was shown in refs. [7, 8], they lead to non-trivial quantum corrections to the free hypermultiplet action, which modify the kinetic terms and generate a non-trivial scalar potential of matter. In all these studies, HSS appears to be the indispensable tool for quantum calculations, since the very transparent and practical $N = 2$ Feynman rules can be formulated only in HSS. Though the HSS method assumes the infinite number of ghosts and auxiliary fields in

components, it does not really lead to complications in interpreting the results that can be reformulated in the conventional superspace language or in components (with a finite number of fields involved). The connection between the HSS approach and the more conventional methods is, however, non-trivial and it needs to be developed further.

In sect. 2 the HSS approach is reviewed along the lines of the original papers [2, 3]. It simultaneously introduces our notation. In sect. 3 we complete the calculation of the next-to-leading-order non-holomorphic correction to the gauge LEEA in the $N = 2$ super-QED, which was initiated in refs. [4, 5], and then generalize it to the case of the spontaneously broken $N = 2$ *super-Yang-Mills* (SYM) theory, including the Seiberg-Witten model, and add the fundamental hypermultiplet matter as well. In sect. 4, the results of sect. 3 are applied to the finite and scale-invariant $N = 2$ supersymmetric gauge theories. The relevance of the perturbative non-holomorphic contributions to the LEEA of finite $N = 2$ supersymmetric gauge field theories in four dimensions for checking M-theory was recently emphasized by Dine and Seiberg in ref. [9].

2   Basic facts about $N = 2$ gauge theories in HSS

In the HSS formalism, the standard N=2 superspace $z = (x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha} i})$, where $m = 0, 1, 2, 3$, $\alpha = 1, 2$, and $i = 1, 2$, is extended by adding the bosonic variables (or 'zweibeins') $u^{\pm i}$ parametrizing the sphere $S^2 \sim SU(2)/U(1)$. The $SU(2)$ indices are raised and lowered with the antisymmetric Levi-Civita symbols ε_{ij} and ε^{ij}, $\varepsilon^{12} = -\varepsilon_{12} = 1$. The ordinary complex conjugation is detoned by bar. One has

$$\begin{pmatrix} u^{+i} \\ u^{-i} \end{pmatrix} \in SU(2) \,, \quad \text{so that} \quad u^{+i} u_i^- = 1 \,, \quad \text{and} \quad u^{+i} u_i^+ = u^{-i} u_i^- = 0 \,. \quad (2.1)$$

Instead of employing an explicit parametrization of the sphere, it is convenient to deal with functions of zweibeins, that carry a definite $U(1)$ charge q to be defined by $q(u_i^\pm) = \pm 1$, and use the following integration rules [2]:

$$\int du = 1 \,, \qquad \int du \, u^{+(i_1} \cdots u^{+i_m} u^{-j_1} \cdots u^{-j_n)} = 0 \,, \quad \text{when} \quad m + n > 0 \,. \quad (2.2)$$

It is obvious that the integral over a $U(1)$-charged quantity vanishes.

In addition to the usual complex conjugation, there exists a star conjugation that only acts on the $U(1)$ indices, $(u_i^+)^* = u_i^-$ and $(u_i^-)^* = -u_i^+$. One easily finds [2]

$$\overline{\overset{*}{u^{\pm i}}} = -u_i^\pm \,, \qquad \overline{\overset{*}{u_i^{\pm}}} = u^{\pm i} \,. \quad (2.3)$$

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The covariant derivatives with respect to the zweibeins, that preserve the defining conditions (2.1), are given by

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} , \quad D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}} , \quad D^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} . \tag{2.4}$$

It is easy to check that they satisfy the $SU(2)$ algebra,

$$[D^{++}, D^{--}] = D^0 , \quad [D^0, D^{\pm\pm}] = \pm 2 D^{\pm\pm} . \tag{2.5}$$

The key feature of the $N = 2$ HSS is the existence of the *analytic* subspace parametrized by the coordinates

$$(\zeta, u) = \left\{ x_A^m = x^m - 2i\theta^{(i} \sigma^m \bar{\theta}^{j)} u_i^+ u_j^- , \quad \theta_\alpha^+ = \theta_\alpha^i u_i^+ , \quad \bar{\theta}_{\dot{\alpha}}^+ = \bar{\theta}_{\dot{\alpha}}^i u_i^+ , \quad u_i^\pm \right\} , \tag{2.6}$$

which is invariant under $N = 2$ supersymmetry, and is closed under the combined conjugation of eq. (2.3) [2]. It allows one to define the *analytic* superfields of any $U(1)$ charge q, by the analyticity conditions

$$D_\alpha^+ \phi^{(q)} = \bar{D}_{\dot{\alpha}}^+ \phi^{(q)} = 0 , \quad \text{where} \quad D_\alpha^+ = D_\alpha^i u_i^+ \quad \text{and} \quad \bar{D}_{\dot{\alpha}}^+ = \bar{D}_{\dot{\alpha}}^i u_i^+ , \tag{2.7}$$

and introduce the analytic measure $d\zeta^{(-4)} du \equiv d^4 x_A d^2\theta^+ d^2\bar{\theta}^+ du$ of charge (-4), so that the full measure in the $N = 2$ HSS can be written down as

$$d^4 x d^4\theta d^4\bar{\theta} du = d\zeta^{(-4)} du (D^+)^4 , \tag{2.8}$$

where

$$(D^+)^4 = \frac{1}{16}(D^+)^2 (\bar{D}^+)^2 = \frac{1}{16}(D^{+\alpha} D_\alpha^+)(\bar{D}_{\dot{\alpha}}^+ \bar{D}^{+\dot{\alpha}}) . \tag{2.9}$$

In the analytic subspace, the harmonic derivative

$$D_A^{++} = D^{++} - 2i\theta^+ \sigma^m \bar{\theta}^+ \partial_m \tag{2.10}$$

preserves analyticity and allows one to integrate by parts. Since both the original (central) basis and the analytic one can be used on equal footing in HSS, in what follows we omit the subscript A at the covariant derivatives in the analytic basis.

It is the advantage of the analytic HSS that both a (massless) hypermultiplet and an $N = 2$ vector multiplet can be introduced there on equal footing. Namely, the hypermultiplet can be defined as an unconstrained complex analytic superfield q^+ of the $U(1)$-charge $(+1)$, whereas the $N = 2$ vector multiplet is described by an unconstrained analytic superfield V^{++} of the $U(1)$-charge $(+2)$. The V^{++} is real in the sense $\overline{V^{++}}^* = V^{++}$, and it can be naturally introduced as a connection to the harmonic derivative D^{++}. The both fields, q^+ and V^{++}, can be Lie algebra-valued

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in a fundamental or adjoint representation of the gauge group. The hypermultiplet action with a minimal coupling to the gauge superfield reads

$$S[q, V] = -\text{tr} \int d\zeta^{(-4)} du \, \overset{*}{\overline{q}}{}^+ (D^{++} + iV^{++}) q^+ \ . \tag{2.11}$$

It is not difficult to check that the free hypermultiplet equations of motion, $D^{++}q^+ = 0$ imply $q^+ = q^i(z)u_i^+$ and the usual (on-shell) Fayet-Sohnius constraints [10] in the ordinary $N = 2$ superspace,

$$D_\alpha^{(i} q^{j)}(z) = D_{\dot\alpha}^{(i} q^{j)}(z) = 0 \ . \tag{2.12}$$

There exists another, equivalent HSS description of a massless hypermultiplet in terms of an unconstrained analytic superfield ω with the vanishing $U(1)$-charge [2], and the action

$$S[\omega, V] = -\tfrac{1}{2}\text{tr} \int d\zeta^{(-4)} du \, (D^{++} + iV^{++})\omega(D^{++} + iV^{++})\omega \ . \tag{2.13}$$

The corresponding (V-dependent) effective actions to be obtained by integrating over the hypermultiplet q^+ or ω, respectively, are the same when one trades each q hypermultiplet for a real ω hypermultiplet [4]. The off-shell HSS hypermultiplet in terms of the ordinary $N = 2$ superfields is just the infinitely-relaxed $N = 2$ tensor multiplet [11].

The $N = 2$ SYM theory is usually formulated in the ordinary $N = 2$ superspace by imposing certain constraints on the gauge- and super- covariant derivatives $\mathcal{D}_\alpha^i$ and $\bar{\mathcal{D}}_{\dot\alpha}^i$ [12]. The constraints [12], in essence, boil down to the existence of a covariantly chiral and gauge-covariant $N = 2$ SYM field strength W satisfying the reality condition (Bianchi 'identity')

$$\mathcal{D}^\alpha{}_{(i}\mathcal{D}_{j)\alpha}W = \bar{\mathcal{D}}_{\dot\alpha(i}\bar{\mathcal{D}}^{\dot\alpha}{}_{j)}\bar{W} \ . \tag{2.14}$$

Unlike the $N = 1$ SYM theory, an $N = 2$ supersymmetric solution to the non-abelian $N = 2$ SYM constraints in the ordinary $N = 2$ superspace is not known in an analytic form. It is the $N = 2$ HSS reformulation of the $N = 2$ SYM theory that makes it possible [2]. The exact non-abelian relation between the constrained, harmonic-independent superfield strength W and the unconstrained analytic superfield V^{++} is given in refs. [2, 3], and it is highly non-linear. It is merely its abelian version that is needed for calculating the perturbative LEEA. It is not difficult to check, or just use the results of ref. [13], that the abelian relation takes the form

$$W = \tfrac{1}{4}\{\bar{\mathcal{D}}_{\dot\alpha}^+, \bar{\mathcal{D}}^{-\dot\alpha}\} = -\tfrac{1}{4}(\bar{D}^+)^2 A^{--} \ , \tag{2.15}$$

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where the non-analytic harmonic superfield connection $A^{--}(z,u)$ to the derivative D^{--} has been introduced, $\mathcal{D}^{--} = D^{--} + iA^{--}$. As a consequence of the $N=2$ HSS abelian constraint $[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0 = D^0$, the connection A^{--} satisfies the relation

$$D^{++}A^{--} = D^{--}V^{++} , \tag{2.16}$$

whereas eq. (2.14) can be rewritten to the form

$$(D^+)^2 W = (\bar{D}^+)^2 \bar{W} . \tag{2.17}$$

A solution to the A^{--} in terms of the analytic unconstrained superfield V^{++} easily follows from eq. (2.16) when using the identity [3]

$$D_1^{++}(u_1^+ u_2^+)^{-2} = D_1^{--}\delta^{(2,-2)}(u_1, u_2) , \tag{2.18}$$

where we have introduced the harmonic delta-function $\delta^{(2,-2)}(u_1, u_2)$ and the harmonic distribution $(u_1^+ u_2^+)^{-2}$ according to their definitions in refs. [2, 3], hopefully, in the self-explaining notation. One finds [13]

$$A^{--}(z,u) = \int dv \, \frac{V^{++}(z,v)}{(u^+ v^+)^2} , \tag{2.19}$$

and

$$W(z) = -\frac{1}{4}\int du (\bar{D}^-)^2 V^{++}(z,u) , \quad \bar{W}(z) = -\frac{1}{4}\int du (D^-)^2 V^{++}(z,u) , \tag{2.20}$$

by using the identity

$$u_i^+ = v_i^+(v^- u^+) - v_i^-(u^+ v^+) , \tag{2.21}$$

which is the obvious consequence of the definitions (2.1).

The equations of motion are given by the vanishing analytic superfield

$$(D^+)^4 A^{--}(z,u) = 0 , \tag{2.22}$$

while the corresponding action reads [13]

$$\begin{aligned} S[V] &= \frac{1}{4}\int d^4x d^4\theta \, W^2 + \text{h.c.} = \frac{1}{2}\int d^4x d^4\theta d^4\bar{\theta} du \, V^{++}(z,u) A^{--}(z,u) \\ &= \frac{1}{2}\int d^4x d^4\theta d^4\bar{\theta} du_1 du_2 \, \frac{V^{++}(z,u_1) V^{++}(z,u_2)}{(u_1^+ u_2^+)^2} . \end{aligned} \tag{2.23}$$

In a WZ-like gauge, the abelian analytic pre-potential V^{++} amounts to [2]

$$\begin{aligned} V^{++}(x_A, \theta^+, \bar{\theta}^+, u) =& \bar{\theta}^+ \bar{\theta}^+ a(x_A) + \bar{a}(x_A)\theta^+\theta^+ - 2i\theta^+\sigma^m\bar{\theta}^+ V_m(x_A) \\ &+ \bar{\theta}^+\bar{\theta}^+\theta^{\alpha+}\psi_\alpha^i(x_A)u_i^- + \theta^+\theta^+\bar{\theta}_{\dot{\alpha}}^+\bar{\psi}^{\dot{\alpha}i}(x_A)u_i^- \\ &+ \theta^+\theta^+\bar{\theta}^+\bar{\theta}^+ D^{(ij)}(x_A)u_i^- u_j^- , \end{aligned} \tag{2.24}$$

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where $(a, \psi_\alpha^i, V_m, D^{ij})$ are the usual $N = 2$ vector multiplet components [12].

The (BPS) mass of a hypermultiplet can only come from the central charges of the $N = 2$ SUSY algebra since, otherwise, the number of the massive hypermultiplet components has to be increased. The most natural way to introduce central charges $(Z, \bar{Z})$ is to identify them with spontaneously broken $U(1)$ generators of dimensional reduction from six dimensions via the Scherk-Schwarz mechanism [14]. Being rewritten to six dimensions, eq. (2.10) implies the additional 'connection' term in the associated four-dimensional harmonic derivative

$$\mathcal{D}^{++} = D^{++} + v^{++} , \quad \text{where} \quad v^{++} = i(\theta^+\theta^+)\bar{Z} + i(\bar{\theta}^+\bar{\theta}^+)Z . \tag{2.25}$$

Comparing eq. (2.25) with eqs. (2.11) and (2.20) clearly shows that the $N = 2$ central charges can be equivalently treated as a non-trivial $N = 2$ gauge background, with the covariantly constant chiral superfield strength

$$\langle W \rangle = \langle a \rangle = Z , \tag{2.26}$$

where eq. (2.24) has been used too.

3 The one-loop LEEA for $N = 2$ gauge fields

The gauge-invariant hypermultiplet action in the $N = 2$ SYM background with the gauge group $SU(2)$ is to be supplemented by a gauge-fixing term and the corresponding ghost terms. The $N = 2$ ghost structure within the background-field method in HSS was recently studied in ref. [5]. As was shown in ref. [5] there are two types of ghosts in the adjoint representation of $SU(2)$: the Faddeev-Popov (FP) fermionic ghosts to be represented by two real ω_{FP}-hypermultiplets, and the Nielsen-Kallosh (NK) bosonic ghosts to be represented by a real ω_{NK}-hypermultiplet. Most importantly, the $N = 2$ SYM one-loop effective action in HSS is found to be entirely determined by the ghost contributions alone [5].

In the Coulomb branch of the quantum theory, the gauge group $SU(2)$ is broken to its $U(1)$ subgroup so that only an abelian $N = 2$ gauge component represents the light degrees of freedom at a generic point in the moduli space of vacua, [3]

$$V^{++} \equiv \frac{1}{\sqrt{2}}\tau^a V^{a++} \rightarrow \frac{1}{\sqrt{2}}\tau^3 V^{3++} , \tag{3.1}$$

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[3]We use the normalization condition $\text{tr}(t^a t^b) = \delta^{ab}$ for the $SU(2)$ generators t^a, $a = 1, 2, 3$, in *any* representation. The $SU(2)$ gauge coupling constant is set to be $e^2 = 2$.
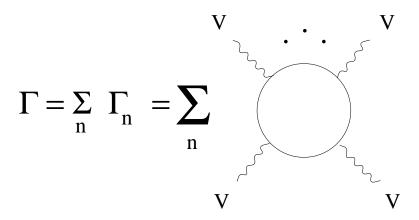
Fig. 1. A perturbation series for the abelian LEEA.

where Pauli matrices τ^a satisfy the relations $[\tau^a, \tau^b] = 2i\varepsilon^{abc}\tau^c$ and $\mathrm{tr}(\tau^a\tau^b) = 2\delta^{ab}$. When being only interested in calculating the leading perturbative correction to the effective action, we do not have to integrate over the massive gauge fields, so that they can be simply dropped out of the microscopic Lagrangian. A real hypermultiplet ω of unit charge, in the adjoint of $SU(2)$ then yields a *complex* hypermultiplet of charge $\sqrt{2}$, minimally coupled to the $U(1)$ gauge field V^{3++}, since one of the hypermultiplet components (ω^3) decouples. Therefore, the total purely $N = 2$ SYM contribution to the LEEA is given by *minus* that of a real bosonic ω-hypermultiplet in the adjoint or, equivalently, by *minus twice* that of a single bosonic q-hypermultiplet of charge $\sqrt{2}$ [5]. Similarly, the LEEA contribution of a matter q-hypermultiplet with unit charge, in the fundamental representation of $SU(2)$ is *twice* that of a single q-hypermultiplet of charge $1/\sqrt{2}$.

Having reduced a calculation of the perturbative LEEA in the gauge sector to the *abelian* problem for a single q-hypermultiplet minimally coupled to the background $U(1)$ gauge superfield V^{++} in HSS, we now have to integrate over the q-hypermultiplet. The corresponding basic effective action $\Gamma_q[V]$ is given by a sum of the one-loop HSS graphs in powers of the background gauge superfield, and the loop to be constructed out of the hypermultiplet propagators (Fig. 1).

It is straightforward to calculate each HSS graph, by using either the massive q-hypermultiplet propagator with the BPS mass [8],

$$i\left\langle q^+(1)\,\overset{*}{\bar{q}}{}^+(2)\right\rangle = -\frac{1}{\Box_1 + Z\bar{Z}}(D_1^+)^4(D_2^+)^4 e^{v_2-v_1}\delta^{12}(z_1 - z_2)\frac{1}{(u_1^+u_2^+)^3}\,,\qquad (3.2a)$$

where

$$v \equiv i(\theta^+\theta^-)\bar{Z} + i(\bar{\theta}^+\bar{\theta}^-)Z\,,\qquad\qquad (3.2b)$$

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or, equivalently, the massless one ($Z = v = 0$) but with the special background gauge superfield, $iV^{++} \to iV^{++} + v^{++}$, where v^{++} is given by eq. (2.25).

The calculation goes as follows [4]. First, one restores the full Grassmann measure by taking the factors $(D_1^+)^4 \cdots (D_n^+)^4$ off the hypermultiplet propagators. It allows one to explicitly integrate over all but one set of the anticommuting HSS coordinates by using the Grassmann delta-functions, thus obtaining the full Grassmann measure $d^8\theta$. Further integrating by parts and using eq. (2.21), one can cancel the harmonic distribution $(u_1^+ u_2^+)^{-3}(u_2^+ u_3^+)^{-3} \cdots (u_{n-1}^+ u_n^+)^{-3}(u_n^+ u_1^+)^{-3}$, which simultaneously means the absence of potential divergences in the harmonic variables, in accordance with the general analysis of ref. [3]. Because of the gauge invariance, the remaining local terms in the low-energy approximation can only depend upon W and $\bar{W}$ via eq. (2.20). It allows one to eliminate all the dependence upon the harmonic variables in the LEEA, and end up with

$$\Gamma[V] = \left[\int d^4x \, d^4\theta \, \mathcal{F}(W) + \text{h.c.} \right] + \int d^4x \, d^4\theta \, d^4\bar{\theta} \, \mathcal{H}(W, \bar{W}) \ . \tag{3.3}$$

The holomorphic contribution to the one-loop LEEA appears due to the non-vanishing central charges whose presence gives rise to the $(\theta^-)^4$-dependent terms before Grassmann integration in the HSS graphs. As was shown in ref. [4], these terms deliver Seiberg's perturbative LEEA [6],

$$\mathcal{F}_q(W) = -\frac{1}{32\pi^2} W^2 \ln \frac{W^2}{M^2} \ , \tag{3.4}$$

where the renormalization scale M is fixed by the condition $\mathcal{F}_q(M) = 0$. Note that the result (3.4) does not depend upon an infra-red cutoff Λ.

Similarly, the non-holomorphic perturbative contribution from the one-loop HSS graphs is given by [4]

$$\mathcal{H}_q(W, \overline{W}) = \frac{1}{(16\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left(\frac{W\overline{W}}{\Lambda^2} \right)^k = \frac{1}{(16\pi)^2} \int_0^{W\overline{W}/\Lambda^2} \frac{d\xi}{\xi} \ln(1 + \xi) \ , \tag{3.5}$$

where we have used the standard integral representation for the dilogarithm function. The first term in eq. (3.5) was calculated in ref. [4], where it was interpreted as the $N = 2$ supersymmetric Heisenberg-Euler Lagrangian.

It is not difficult to verify that the asymptotical perturbation series (3.5) can be rewritten to the form suggested for the abelian case by de Wit, Grisaru and Roček in ref. [15],

$$\mathcal{H}_q(W, \overline{W}) = \frac{1}{(16\pi)^2} \ln \left(\frac{W}{\Lambda} \right) \ln \left(\frac{\overline{W}}{\Lambda} \right) \ , \tag{3.6}$$

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[4] The infra-red cutoff Λ is fixed by the condition $\mathcal{H}_q(\Lambda, \Lambda) = 0$.

or, equivalently, as (see eq. (4.4) below)

$$\mathcal{H}_q(W, \overline{W}) = \frac{1}{2(16\pi)^2} \ln^2\left(\frac{W\overline{W}}{\Lambda^2}\right) , \tag{3.7}$$

where we have used the fact that the non-holomorphic function $\mathcal{H}$ is defined modulo the Kähler gauge transformations

$$\mathcal{H}(W, \overline{W}) \rightarrow \mathcal{H}(W, \overline{W}) + f(W) + \bar{f}(\overline{W}) , \tag{3.8}$$

with an arbitrary holomorphic function $f(W)$ as a parameter. The non-holomorphic contribution of eq. (3.6) or (3.7) does not really depend upon the scale Λ, again due to the Kähler invariance (3.8).

The HSS result (3.4) for the perturbative part of the holomorphic (SW) $N = 2$ gauge LEEA agrees with the Seiberg argument [6] based on the perturbative $U(1)_R$ symmetry and an integration of the associated chiral anomaly. As is obvious from eq. (3.6), the next-to-leading-order non-holomorphic contribution to the SW gauge LEEA satisfies a simple differential equation

$$W\overline{W}\partial_W\partial_{\overline{W}}\mathcal{H}_q(W, \overline{W}) = const. , \tag{3.9}$$

which can be considered as the direct consequence of scale and $U(1)_R$ invariances [9].

Since the Seiberg result [6] for the perturbative holomorphic function $\mathcal{F}(W)$ was indirectly obtained by integrating the chiral anomaly, its direct and manifestly $N = 2$ supersymmetric HSS derivation [4] means, in particular, that the one-loop contribution (3.4) *exactly* saturates the corresponding anomalous perturbative LEEA.

It is natural to replace Λ in eq. (3.6) with a field-dependent cutoff a whose vacuum expectation value $\langle a \rangle$ parameterizes the moduli space of vacua, and can be identified with the central charge because of eq. (2.26),

$$\mathcal{H}_q(W, \bar{W}) = \frac{1}{2(16\pi)^2} \ln^2\left(\frac{W\bar{W}}{a\bar{a}}\right) . \tag{3.10}$$

As a simple application, consider the celebrated Seiberg-Witten model, whose fundamental (microscopic) action describes the purely gauge $N = 2$ SYM theory, with the $SU(2)$ gauge group spontaneously broken to its $U(1)$ subgroup [1]. The perturbative low-energy effective action reads

$$\begin{aligned}
\Gamma^{\text{SW}}_{\text{perturbative}} &= -4 \int_{\text{chiral}} \mathcal{F}_q(W) - 2 \int \mathcal{H}_q(W, \bar{W}) \\
&= +\frac{1}{(4\pi)^2} \int_{\text{chiral}} W^2 \ln\frac{W^2}{M^2} - \frac{1}{(16\pi)^2} \int \ln^2\left(\frac{W\bar{W}}{a\bar{a}}\right) .
\end{aligned} \tag{3.11}$$

In the strong coupling region, near a singularity in the quantum moduli space where a BPS-like (t'Hooft-Polyakov) monopole becomes massless, the Seiberg-Witten model takes the form of the *dual* $N = 2$ supersymmetric QED after the duality transformation $V^{++} \to V_D^{++}$, $W \to W_D$ and $a \to a_D$. The t'Hooft-Polyakov monopole is known to belong to a q^+-hypermultiplet that represents the non-perturbative degrees of freedom in the theory [1]. Therefore, the low-energy effective action near the monopole singularity is given by

$$
\begin{aligned}
\Gamma_D^{\text{SW}} &= + \int_{\text{chiral}} \mathcal{F}_q(W_D) + \int \mathcal{H}_q(W_D, \bar{W}_D) \\
&= -\frac{1}{32\pi^2} \int_{\text{chiral}} W_D^2 \ln \frac{W_D^2}{M_D^2} + \frac{1}{2(16\pi)^2} \int \ln^2\left(\frac{W_D \bar{W}_D}{a_D \bar{a}_D}\right) .
\end{aligned}
\tag{3.12}
$$

The exact holomorphic low-energy effective action [1] is known to have just two physical singularities in the quantum moduli space, where a BPS-like particle becomes massless. The perturbative next-to-leading-order term $\mathcal{H}(W, \bar{W})$ is not singular at that points. Of course, we still have to justify its use at strong coupling (see the next sect. 4). The $SL(2, \mathbf{Z})$ duality requires the *exact* function $\mathcal{H}(W, \bar{W})$ to be duality-invariant [16]. Being combined with the perturbative information from eqs. (3.11) and (3.12), it is, however, not enough to determine the exact form of that function (see ref. [17] for some additional proposals).

In a more general case of N_f, q^+-type hypermultiplets in the fundamental representation of the gauge group $SU(N_c)$, i.e. the $N = 2$ super-QCD, the extra coefficient in front of the holomorphic contribution $\mathcal{F}$ is proportional to the one-loop RG beta-function $(N_f - 2N_c)$, whereas the extra coefficient in front of the non-holomorphic contribution $\mathcal{H}$ is proportional to $(2N_f - N_c)$, in the $N = 2$ super-Feynman gauge used above. In another interesting case of the $N = 4$ super-Yang-Mills theory, whose $N = 2$ matter content is given by a single ω-type hypermultiplet in the *adjoint* representation of the gauge group, the numerical coefficient in front of the holomorphic function $\mathcal{F}$ vanishes together with the RG beta-function, whereas the numerical coeffient in front of the non-holomorphic contribution $\mathcal{H}$ always appears to be positive, in agreement with the earlier calculations in terms of $N = 1$ superfields (see page 390 of ref. [18]) and some recent $N = 2$ supersymmetric calculations by different methods [19, 20]. In particular, in the case of finite and $N = 2$ superconformally invariant gauge field theories ($N_f = 2N_c$), the leading non-holomorphic contribution to the LEEA is given by eq. (3.5) multiplied by $3N_c$, and it never vanishes.

4   Instanton corrections

The exact solution to the holomorphic LEEA was obtained by Seiberg and Witten [1] by the use of duality and holomorphicity properties of the $N = 2$ SYM theory with the $SU(2)$ gauge group, under certain physical assumptions about the global structure of the quantum moduli space of vacua. The Seiberg-Witten solution is encoded in terms of the auxiliary elliptic curve

$$y^2 = (x^2 - u)^2 - \Lambda_{\mathrm{SW}}^4 \ , \tag{4.1}$$

where the moduli space parameter u can be identified with the expectation value of the gauge-invariant operator, $u = \langle \mathrm{tr}\, a^2 \rangle$, and Λ_{SW} is the renormalisation group invariant (Seiberg-Witten) scale. The holomorphic function $\mathcal{F}(W)$ can then be parametrized in terms of the periods of certain abelian differentials of the 3rd kind, associated with the SW curve (4.1), see e.g., ref. [21] for a review. When being expanded in the inverse powers of W at weak coupling (near $u = \infty$), the Seiberg-Witten solution reads

$$\mathcal{F}_{\mathrm{per.}}(W) = \frac{1}{32\pi^2} \left[W^2 \ln \frac{W^2 \sqrt{2}}{\Lambda_{\mathrm{SW}}^2} - \frac{(\Lambda_{\mathrm{SW}}/2)^4}{W^2} + \dots \right] \ , \tag{4.2}$$

where the first term coincides with that in eq. (3.11) after identifying $M^2 = \Lambda_{\mathrm{SW}}^2/\sqrt{2}$, whereas the rest of terms represents multi-instanton corrections (the leading one-instanton correction is explicitly written down). The latter can be computed independently [22], in agreement with the exact Seiberg-Witten result. The large-distance instanton effects can be encoded in terms of the effective *Callan-Dashen-Gross* (CDG) vertex to be added to the microscopic Lagrangian [23]. Its $N = 2$ supersymmetric generalization was constructed by Yung [24], who also applied it to explicitly calculate the one-instanton corrections to the effective action from the first principles. When being restricted to the light degrees of freedom which are relevant for the LEEA, the momentum expansion of the Yung vertex confirms eq. (4.2) and also yields the one-instanton correction (in the Pauli-Villars regularization scheme) to the non-holomorphic LEEA of the Seiberg-Witten model [24],

$$\mathcal{H}(W, \bar{W})_{\mathrm{per.+one\ inst.}} = \frac{1}{(8\pi)^2} \left(- \left| \ln \frac{W}{\Lambda_{\mathrm{SW}}} \right|^2 + \frac{\Lambda_{\mathrm{SW}}^4}{2W^4} \ln \frac{\bar{W}}{\Lambda_{\mathrm{SW}}} + \frac{\Lambda_{\mathrm{SW}}^4}{2\bar{W}^4} \ln \frac{W}{\Lambda_{\mathrm{SW}}} + \dots \right) \ , \tag{4.3}$$

where the dots stand for higher multi-instanton corrections. In rewriting eq. (4.3) we have used the identity

$$\ln^2 \left(\frac{W\bar{W}}{\Lambda^2} \right) = 2 \ln \frac{W}{\Lambda} \ln \frac{\bar{W}}{\Lambda} + \ln^2 \frac{W}{\Lambda} + \ln^2 \frac{\bar{W}}{\Lambda} \ , \tag{4.4}$$

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and kept only the non-holomorphic terms because of eq. (3.8). The relative easyness of getting the perturbative and one-instanton corrections to the LEEA versus the multi-instanton ones is related to the fact that the former can be calculated by dropping the heavy gauge fields in the microscopic (CDG-modified) Lagrangian.

Being unable to calculate the exact function $\mathcal{H}(W, \bar{W})$ in the Seiberg-Witten model at strong coupling, we can, nevertheless, ask whether at certain circumstances the one-loop perturbative results could be exact. In the case with the $N = 2$ matter to be represented by N_f hypermultiplets in the fundamental representation of $SU(2)$, the corresponding elliptic curve generalizing that of eq. (4.1) is given by [25, 26] [5]

$$y^2 = (x^2 - u)^2 - \Lambda_{\text{SW}}^{4-N_f} \prod_{j=1}^{N_f} (x - m_j) , \qquad (4.5)$$

where m_j are hypermultiplet masses. The dependence upon Λ disappears at $N_f = 4$, which corresponds to the finite and scale-invariant $N = 2$ gauge theories. In this case, there can be neither higher-loop perturbative corrections since they are dependent upon the normalization scale, [6] nor non-perturbative instanton contributions since they are all proportional to the positive powers of Λ (see also ref. [9], as well as some more checks in ref. [28]). Hence, the one-loop non-holomorphic contribution to the LEEA is exact when $N_f = 4$. In accordance with the results of sect. 3, it is given by

$$\mathcal{H}_{\text{finite}}(W, \bar{W}) = \frac{3}{256\pi^2} \ln^2 \left(\frac{W\bar{W}}{\langle W \rangle \langle \bar{W} \rangle} \right) . \qquad (4.6)$$

Note that eq. (4.6) does not depend upon any scale.

Another interesting limit, where the perturbative results of sect. 3 may be exact is to take $\Lambda \to 0$ at $1 \leq N_f \leq 4$. By tuning the bare hypermultiplet masses, one can arrange the situation when the (singular) points in the Coulomb branch, where some of the non-perturbative states (like monopoles or dyons) become massless, coincide at the so-called Argyres-Douglas point [29]. It should lead to a new physics since these BPS-like physical states are mutually *non-local*. The latter means that there is no duality transformation to another field description of these states where the corresponding fields would have no magnetic charges. As was argued in ref. [30], it yields new non-trivial $N = 2$ superconformally invariant gauge field theories. To control the theory at a non-trivial Argyres-Douglas fixed point, it was important for the analysis of ref. [30] to have a path to the Higgs branch that touches the

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[5]When $2 < N_f \leq 4$, there are some ambiguities in the form of elliptic curve [25, 26].

[6] The absence of higher-loop perturbative corrections to the holomorphic LEEA was verified at two loops in ref. [27].

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Coulomb branch at the phase transition points where some of the hypermultiplets become massless. There is no Higgs branch at $N_f = 0$ or 1, but is appears at $N_f = 2$ or 3. It may not be accidental that the coefficient $(N_f - 1)$ in front of the non-holomorphic term is related to it. The well-known fact that the one-instanton holomorphic correction vanishes when there is a massless matter hypermultiplet, may also be related to the Higgs branch in the full quantum theory.

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